

**DIFFERENTIAL GEOMETRY II MIDTERM
EXAMINATION**

Attempt all questions, total marks 100, 10 am - 1 pm

- (1) Let $\alpha^1, \dots, \alpha^k$ be 1-covectors on a vector space V . Show that $\alpha^1 \wedge \dots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \dots, \alpha^k$ are linearly independent in the dual vector space V^* . (15 marks)
- (2) Prove that the real projective space $\mathbb{R}P^n$ is a smooth manifold (set, topology, smooth atlas) and then show that it is compact. (20 marks)
- (3) Let $X_1, \dots, X_n$ be n smooth vector fields on an open subset U of a smooth manifold M of dimension n . Suppose for $p \in U$, the vectors $(X_1)_p, \dots, (X_n)_p$ are linearly independent. Show that there is a chart $(V, x^1, \dots, x^n)$ on M about p such that $(X_i)_p = (\frac{\partial}{\partial x_i})_p$ for $i = 1, \dots, n$. (15 marks)
- (4) Show that the graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,
$$\Gamma(f) = \{(x, y, f(x, y)) \in \mathbb{R}^3\}$$
is a regular submanifold of $\mathbb{R}^3$. (15 marks)
- (5) Show that a smooth map f from a compact smooth manifold N to $\mathbb{R}^m$ has a critical point. (15 marks)
- (6) Let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^n)$ be overlapping coordinate charts on a manifold M . They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on the total space TM of the tangent bundle with transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$. Compute the Jacobian matrix of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$, and prove that its determinant is $(\det(\frac{\partial y^i}{\partial x^j}))^2$ (evaluated at $\phi(p)$). (20 marks)